

Nonlinear limits to the information capacity of optical fiber communications

Partha P Mitra

Jason B Stark

Bell Laboratories, Lucent Technologies, Murray Hill, NJ 07974

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The exponential growth in the rate at which information can be communicated through an optical fiber is a key element in the so called information revolution. However, like all exponential growth laws, there are physical limits to be considered. The nonlinear nature of the propagation of light in optical fiber has made these limits difficult to elucidate. Here we obtain basic insights into the limits to the information capacity of an optical fiber arising from these nonlinearities. The key simplification lies in relating the nonlinear channel to a linear channel with multiplicative noise, for which we are able to obtain analytical results. In fundamental distinction to the linear additive noise case, the capacity does not grow indefinitely with increasing signal power, but has a maximal value. The ideas presented here have broader implications for other nonlinear information channels, such as those involved in sensory transduction in neurobiology. These have been often examined using additive noise linear channel models, and as we show here, **nonlinearities can change the picture qualitatively.**

The classical theory of communications *à la* Shannon [1] was developed mostly in the context of linear channels with additive noise, which was adequate for electromagnetic propagation through wires and cables that have until recently been the main conduits for information flow. Fading channels or channels with multiplicative noise have been considered, for example in the context of wireless communications [2], although such channels remain theoretically less tractable than the additive noise channels. However, with the advent of optical fiber communications we are faced with a nonlinear propagation channel that poses major challenges to our understanding. The difficulty resides in the fact that the input output relationship of an optical fiber channel is obtained by integrating a nonlinear partial differential equation and may not be represented by an instantaneous nonlinearity. Channels where the nonlinearities in the input output relationship are not instantaneous are in general ill understood, the optical fiber simply being a case of current relevance. The understanding of such nonlinear channels with memory are of fundamental interest, both because communication rates through optical fiber are increasing exponentially and we need to know where the limits are, and also because understanding such channels may give us insight elsewhere, such as into the design principles of neurobiological information channels at the sensory periphery.

The capacity of a communication channel is the maximal rate at which information may be transferred through the channel without error. The capacity can be written as a product of two conceptually distinct quantities, the spectral bandwidth W and the maximal spectral efficiency

which we will denote \mathcal{C} . In the classic capacity formula for the additive white Gaussian noise channel with an average power constraint, $C = W \log(1 + S/N)$ [1], the spectral bandwidth W , which has dimensions of inverse time, multiplies the dimensionless maximal spectral efficiency $\mathcal{C} = \log(1 + S/N)$. Here S and N are the signal and noise powers respectively. It is instructive to examine this formula in the context of an optical fiber. Since the maximal spectral efficiency is logarithmic in the signal to noise ratio (SNR), it can never be too large in a realistic situation, so that the capacity is principally determined by the bandwidth W . In the case of an optical fiber, the intrinsic loss mechanisms of light propagating through silica fundamentally limits W to a maximum of about $50THz$ [3] corresponding to a wavelength range of about $400nm$ ($1.2-1.6\mu$). This is to be compared with current systems where the total bandwidth is limited to about $15THz$. If the channel was linear, the maximal spectral efficiency would be $\mathcal{C} = \log(1 + S/N)$, S being input light intensity and N the intensity of amplified spontaneous emission noise in the system. An output SNR of say 100 (i.e. 20dB), would then yield a spectral efficiency of 6.6, which for a $50THz$ channel would correspond to a capacity of $330Tbit/sec$. The channel, of course, is not linear; how do the nonlinearities impact the spectral efficiency of the fiber? The basic conclusion of the present work is that the impact is severe and qualitative. As shown in Fig.1, the effect is a saturation and eventual decline of spectral efficiency as a function of input signal power, in complete contrast with the linear channel case. We now proceed to motivate and discuss this result.

It is widely recognised that nonlinearities impair the channel capacity. However, estimation of the impact of the nonlinearities on channel capacity has remained *ad hoc* from an information theory perspective. Here we obtain what appears to be the first systematic estimates (Fig.1) for the maximal spectral efficiency of an optical fiber channel as a function of the relevant parameters. In basic distinction to the linear channel, our considerations indicate that the maximal spectral efficiency does not grow indefinitely with signal power, but reaches a maximum of several bits and eventually *declines*, as illustrated in Figure 1. It is to be noted that current systems use a binary signalling scheme which limits the achievable spectral efficiency *a priori* to 1 bit, and to reach the higher spectral efficiencies predicted by the theory, multi-bit signalling schemes would have to be used. Since the spectral efficiencies of current systems are already approaching 1 bit, it is clear that the limits discussed here will be of practical relevance in the future.

Although a number of nonlinearities are present in light propagation in a fiber, we concentrate on the most important one for fiber communications, namely the dependence of the refractive index (and therefore the propagation velocity of light) on the light intensity, $n = n_0 + n_2 I$. This nonlinearity is weak, but its effects accumulate due to the long propagation distances involved in fibre communications, and is responsible for the effects considered here. Three principle physical parameters characterising the propagation are of interest: the group velocity dispersion $\beta \sim 10ps^2/km$, the propagation loss $\alpha \sim 0.2 dB/km$ and the strength of the nonlinear refractive index, usually expressed in terms of the parameter $\gamma \sim 1/W/km$. The propagation loss is compensated by interposing optical amplifiers into the system. Each amplifier also injects spontaneous emission noise into the system with strength $I_1 = aGh\nu\Delta\nu$ [5], with G being the amplifier gain, h the Planck's constant, ν and $\Delta\nu$ being the centre frequency and frequency bandwidth of light respectively. Here 'a' is a numerical constant (which we assume to be 2). For n_s spans of fiber interspersed with amplifiers that make the total channel gain unity, the effects of absorption may be accounted for simply by redefining the system length in terms of an effective length, $L_{eff} \sim n_s/\alpha$. If the nonlinearity were absent ($\gamma = 0$), we would have obtained, for the maximal spectral efficiency, $\mathcal{C}_0 = \log(1 + I/I_n)$, I being the input power and $I_n = n_s I_1$ being the total additive noise power. Note that \mathcal{C}_0 declines logarithmically with system length, and would eventually vanish for infinitely long systems. Note also that although spectral efficiency is dimensionless, it is often written for convenience with the "units" bits/sec/Hz.

For a variety of reasons, the principal one being limitations in the electronic bandwidth, it

is impractical to modulate the full optical bandwidth at once. Instead, current attempts towards achieving maximal information throughput involve so called Wavelength Division Multiplexing (WDM) [3], where the whole optical bandwidth is broken up into disjoint frequency bands (“channels”) each of which is modulated separately. We confine our attention to such systems (which from an information theory perspective corresponds to the “multi-user” case) [6], though we also comment on the ideal case of utilising the full optical bandwidth for a single data stream (the “single user” case). Quantitatively, the single user case is expected to have larger maximal spectral efficiencies, though we will argue that it shows the same qualitative behaviour as the multi-user case. The difference between the two reside in the fact that in the multi-user case, each channel is an independent information stream, and appears as an additional source of noise to every other channel due to nonlinear mixing.

The nonlinear propagation effects in the evolution of the electric field amplitude involve a cubic term in the electric field. In a WDM system, the nonlinearities are classified by the field amplitudes participating in this cubic term for the evolution of the field amplitude of a given channel: self phase modulation denotes the case where all three fields belong to the same channel, cross phase modulation where two fields belong to a different channel and one to the same channel, and four wave mixing denotes the case where all three amplitudes belong to different channels. Out of these terms, four wave mixing gives rise to additive noise to the channel of interest and will not be considered further in this paper. One reason for this is that four wave mixing is strongly suppressed by dispersion when the channel spacings are substantial. Its effects can be accounted for by augmenting the additive noise term in the subsequent considerations. We also neglect self phase modulation effects, since these effects are deterministic for the given channel and in principle could be reduced by using nonlinear precompensation. Finally, we are left with cross phase modulation, which appears to be the principle source of nonlinear capacity impairment in the multiuser case for realistic parameter ranges. A further reason for our focus on cross phase modulation is that it gives rise to multiplicative noise, which gives rise to qualitatively new effects in the channel capacity.

We model the propagation channel in the presence of cross phase modulation by means of a *linear Schroedinger equation* with a *random potential* fluctuating both in space and time. This is easily justified starting from the nonlinear Schroedinger equation description commonly used to describe light propagation in single mode optical fibres [4]. Cross phase modulation arises from terms in the equation where the field intensity in the nonlinear refractive index is approximated by the sum of the field intensities in the channels other than the one for which the propagation is being studied. Therefore, if only cross phase modulation effects were retained, the propagation equation for the field amplitude in channel i then becomes

$$i\partial_z E_i = \frac{\beta_2}{2} \partial_t^2 E_i + V(z, t) E_i, \quad (1)$$

where $V(z, t) = -2\gamma \sum_{j \neq i} |E_j(z, t)|^2$, the sum being taken over the other channels. Since independent streams of information are transmitted in the other channels, $V(z, t)$ appears as a random noise term. Notice that the nonlinear propagation equation has now been reduced to a linear Schroedinger equation with a stochastic potential, so that the nonlinear channel has become a channel with multiplicative noise. We now need an adequate model for the stochastic properties of $V(z, t)$. If the dispersion is substantial, we propose that $V(z, t)$ may be approximated by a Gaussian stochastic process short range correlated in both space and time. Since V is obtained by adding a large number of different channels, each of which is short range correlated in time ($\tau \sim 1/B$, where B is the channel bandwidth), we can expect V to have a correlation time of approximately $1/B$. Dispersion causes the channels to travel at different speeds, thus causing V to be short range correlated in space as well, with a correlation length related to the dispersion length. Since V is a sum of intensities, it has nonzero mean, so we define $\delta V(z, t) = V(z, t) - \langle V \rangle$, where $\langle V \rangle$ denotes the average value of V . Removing a

constant from the potential causes an overall phase shift independent of space and time, which is irrelevant to the present considerations.

The parameter of interest in the following is the integrated strength of the fluctuating field, $\eta = \int dz \langle \delta V(z, 0) \delta V(0, 0) \rangle$. In order to estimate η , we consider a simplified propagation model for the channels other than the one of interest, in which nonlinearities are neglected, and stochastic bit streams at the inputs to the channels are propagated forward with constant group velocities. The group velocity difference between two channels separated by a spacing $\Delta\lambda$ is $D\Delta\lambda$. In this model with n_c other channels evenly spaced by $\Delta\lambda$ around the channel of interest, each with intensity I and bandwidth B , we obtain $\eta = 2 \ln(n_c/2)(\gamma I)^2 / (BD\Delta\lambda)$. Here D is the dispersion parameter $D = -2\pi c\beta/\lambda^2$. Although this is a simplified model for the other channels, numerical simulations of propagation including the nonlinearities and dispersion for the side channels show that the estimate of η is accurate.

Note that the denominator in the expression of η is the inverse of the dispersion length L_D for the given channel spacing. This form for η follows from assuming that $L_{eff} \gg L_D$, since in this limit the integral defining η is cut off by L_D . If on the other hand, $L_{eff} \leq L_D$, the integral would be cut off by L_{eff} , so that one would have to replace L_D by L_{eff} in the equation for η . The fluctuation strength scales with the logarithm of the number of channels rather than the total number since channels at larger spacings are suppressed proportionately to channel spacing. This suppression due to dispersion leads to the logarithmic factor via a sum of the form $\sum_j 1/(\Delta\lambda_j) \propto \sum_j 1/j$.

Within the model under consideration, the propagation down the fiber is given in terms of a propagator $U(t, t'; L)$ obtained by integrating the stochastic Schroedinger equation. For simplicity, we model the amplifier noise as an additive term with strength I_n as defined earlier. The channel is specified in terms of a relation between the input and output electric field amplitudes, $E_{out}(t) = \int dt' U(t, t'; L) E_{in}(t') + n(t)$. Since U is stochastic, due to the underlying stochasticity of $V(z, t)$, the model corresponds to a channel with multiplicative noise. It is still intractable in terms of an exact capacity computation, but an analytic lower bound may now be obtained. This bound is based on the following information theoretic result (E.Telatar, private communications): the capacity \mathcal{C} of a channel with input X and output Y related by a conditional distribution $p(Y|X)$ and an input power constraint $E(\|X\|^2) = P$ satisfies the inequalities $\mathcal{C} = \max_{p(X)} I(X, Y) \geq I(X_G, Y) \geq I(X_G, Y_G)$ Here $I(X_G, Y)$ is the mutual information when $p(X)$ is chosen to be $p_G(X)$, a Gaussian satisfying the power constraint; $I(X_G, Y_G)$ is the mutual information of a pair (X_G, Y_G) with the same second moments as the pair (X, Y) . The first inequality is trivial since $p_G(X)$ is not necessarily the optimal input distribution. A proof of the second inequality is outlined in the methods section.

The quantity $I(X_G, Y_G)$ for the channel defined above may be computed from knowledge of the correlators $\langle E_{in}(t) E_{in}^*(t') \rangle$, $\langle E_{out}(t) E_{out}^*(t') \rangle$ and $\langle E_{out}(t) E_{in}^*(t') \rangle$. The first is defined *a priori* through the assumption of bandlimited Gaussian white noise input with a power constraint. The second follows from the first using the unitarity of U . The third correlator requires computation of the average propagator $\langle U \rangle$, where the average is over realisations of $V(z, t)$. For a Gaussian, delta-correlated V , we obtain $\langle U(t, t'; L) \rangle = \exp(-\eta L/2) U_0(t - t'; L)$ (see methods), where U_0 is the propagator for $V = 0$. Assembling these results, we finally obtain an analytic expression for a lower bound C_{LB} to the channel capacity of the stochastic Schroedinger equation model:

$$C_{LB} = n_c B \ln \left(1 + \frac{e^{-(\frac{L}{I_0})^2} I}{I_n + (1 - e^{-(\frac{L}{I_0})^2}) I} \right) \quad (2)$$

where I_0 is given by

$$I_0 = \sqrt{\frac{BD\Delta\lambda}{2\gamma^2 \ln(n_c/2) L_{eff}}} \quad (3)$$

The fundamental departure from a linear channel in the above capacity expression is the appearance of an intensity scale I_0 , which governs the onset of nonlinear effects. To obtain an idea about the value of I_0 , consider the parameter values $B = 40GHz$, $D = 20ps/nm/km$, $\Delta\lambda = 1nm$, $\gamma = 1/W/km$, $n_c = 100$, $L_{eff} \approx n_s/\alpha = 100km$. Then $I_0 = 32mW$. Examination of Eq.3 shows that the intensity scale I_0 at which nonlinearities set in shows reasonable dependence on all relevant parameters, namely it increases with increases in the dispersion, the bandwidth and the channel spacing, but decreases with increasing system length and number of channels.

The most striking feature of Eq.2 is that instead of increasing logarithmically with signal intensity like in the linear case, the capacity estimate actually peaks and then declines beyond a certain input intensity. From Eq.2, it is easily derived that the maximum value is given approximately by $C_{max} \approx \frac{2}{3}n_c B \ln(2I_0/I_n)$, the maximum being achieved for an intensity $I_{max} \approx (I_0^2 I_n/2)^{1/3}$. The reason for this behaviour is that if we consider any particular channel, the signal in the other channels appear as noise in the channel of interest, due to the nonlinearities. This ‘noise’ power increases with the ‘signal’ strength, thus causing degradation of the capacity at large ‘signal’ strength. The behaviour of Eq.2 is graphically illustrated in Fig.1, where the spectral efficiency (bits transmitted per second per unit bandwidth) is shown as a function of input power.

It is of interest to note that if the input intensity is kept fixed, the capacity bound declines exponentially with the system length. This is only to be expected, since the correlations of the electric field should decay exponentially due to the fluctuating potential in the propagation equation. On the other hand, the maximal spectral efficiency given by C_{max} declines only logarithmically in system length, in parallel with the behaviour for linear channels. It can therefore be inferred that if the input power was adjusted with system length instead of being kept fixed, the decline of spectral efficiency with system length will be logarithmic.

Finally, we present qualitative arguments as to why the single user case is expected to show the same non-monotonicity of spectral efficiency with the input signal intensities. In the multi-user case, the noise power as effectively generated by cross phase modulation grows as I^3 since it involves three signal photons. In the single user case, the cubic nonlinearity is a deterministic process that does not necessarily degrade channel capacity. However, subleading processes which involve two signal and one spontaneous noise photon still scale superlinearly in signal intensity, as $I^2 I_n$. Therefore, one should still observe the same behaviour of the effective noise power overwhelming the signal at large signal intensities. Thus, we would still expect the spectral efficiency to decline at large input intensity, though not as rapidly in the multi-user (WDM) case.

Methods

Gaussian bound to the channel capacity

Proof of the inequality $I(X_G, Y) \geq I(X_G, Y_G)$: define $p(X, Y)$ as the product $p_G(X)p(Y|X)$, and $p_G(X, Y)$ to be the joint Gaussian distribution having the same second moments as $p(X, Y)$. Also define $p_G(Y)$ to be the corresponding marginal of $p_G(X, Y)$.

$$I(X_G, Y) = \int dX dY p(X, Y) \log\left(\frac{p(X, Y)}{p_G(X)p(Y)}\right) \quad (4)$$

$$= \int dX dY p(X, Y) \left[\log\left(\frac{p_G(X, Y)}{p_G(X)p_G(Y)}\right) - \log\left(\frac{p_G(X, Y)}{p(X, Y)} \frac{p(Y)}{p_G(Y)}\right) \right] \quad (5)$$

$$(6)$$

Since $p(X, Y)$ and $p_G(X, Y)$ share second moments, the first term on the RHS is $I(X_G, Y_G)$. The second term may be simplified using the convexity of the logarithm, $\langle \log(f) \rangle \leq \log(\langle f \rangle)$ to obtain

$$I(X_G, Y) \geq I(X_G, Y_G) - \log\left[\int dX dY p_G(X, Y) \frac{p(Y)}{p_G(Y)}\right] \quad (7)$$

$$\geq I(X_G, Y_G) \quad (8)$$

The second inequality follows by first performing the integral over X , and noting that $\log(\int dY p(Y)) = \log(1) = 0$.

Derivation of the average propagator $\langle U \rangle$:

This can be done by resumming the perturbation series exactly for $\langle U \rangle$, for delta correlated $V(z, t)$. Alternatively, in the path integral formalism [7],

$$\langle U(t, t'; L) \rangle = U_0(t - t'; L) \langle \langle \exp(i \int_0^L dz V(z, t(z))) \rangle \rangle, \quad (9)$$

where the average is taken over V as well as over paths $t(z)$ satisfying $t(0) = t$, $t(L) = t'$. The result in the paper follows by performing the Gaussian average over V . Since $\phi = \int_0^L dz V(z, t(z))$ is a linear combination of Gaussian variables, it is also Gaussian distributed and satisfies $\langle \exp(i\phi) \rangle = \exp(-\langle \phi^2 \rangle / 2)$. The result follows by noting that for delta correlated V , $\langle \phi^2 \rangle$ is a constant given by ηL . The delta correlations need to be treated carefully, this can be done by smearing the delta functions slightly and leads to the definition of η given earlier in the paper.

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Figure Captions

Figure 1. The curves in Fig.1 represent lower bounds to the spectral efficiency for a homogeneous length of fiber for a multi-user WDM system, given analytically by Eq.2. Although the curves represent lower bounds, we argue in the text that the true capacity shows the same qualitative non-monotonic behaviour with respect to input signal powers. The spectral efficiencies displayed in the figure correspond to the capacity per unit bandwidth, $\mathcal{C} = C/(n\delta\nu)$. Here $\delta\nu$ includes both the channel bandwidths and the inter-channel spacing. The parameters used for the figure are $n_c = 100$, $L_{eff} = 100km$, $D = 20ps/nm/km$, $\delta\nu = 1.5B$ where $B = 10GHz$ is the individual channel width. The two continuous curves correspond to $\gamma = 1/W/km$ and $\gamma = 0.1/W/km$, the lower curve corresponding to $\gamma = 1$. The spontaneous noise strength I_n is computed from the formula $I_n = aGh\nu B$ as explained in the text, with $a = 2$, $G = 1000$, $\nu = 200THz$. The dotted curve represents the spectral efficiencies of the corresponding linear channels given by $\gamma = 0$.

