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Consequences of a possible adiabatic transition between $\nu=1/3$ and $\nu=1$ quantum Hall states in a narrow wire

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We consider the possibility of creating an adiabatic transition through a narrow neck, or point contact, between two different quantized Hall states that have the same number of edge modes, such as $\nu=1$ and $\nu=\frac{1}{3}$. We apply both the composite-fermion and Luttinger-liquid formalism to analyze the transition. We suggest that using such adiabatic junctions one could build a dc step-up transformer, where the output voltage is higher than the input. Difficulties standing in the way of an experimental implementation of the adiabatic junction are addressed. [S0163-1829(98)02104-3]

It has long been understood that quantized Hall states with different Hall conductances cannot be connected adiabatically in the interior of a macroscopic two-dimensional electron system. For a pure system, where the quantized Hall states have energy gaps, the boundary between two quantized states must be characterized by a vanishing energy gap, with associated low-energy excitations. In a disordered system there are generally localized low-energy excitations in the interior of a quantized Hall region, which then become extended at the boundary between two quantized regions. The possible transitions between different quantized Hall states have been elucidated (in the case of a fully spin-polarized system) by the introduction of a ''global phase diagram'' based on a unitary transformation which introduces a Chern-Simons gauge field and which, at the mean-field level, maps fractional quantized Hall states onto integer ones.[1,2]

In this Brief Report, we suggest that in a *narrow quantum wire* there can be an adiabatic transition between two different quantized Hall states, under certain conditions. The most important example, to which we restrict ourselves here, is the case of a transition between states with $\nu=1$ and $\frac{1}{3}$. It should be noted that for both these states, there is a single edge mode at a sharp sample boundary,[3] so one can have a single pair of oppositely moving modes running continuously through the transition region. We shall discuss the transition between the two states in a narrow wire using a fermion–Chern-Simons mean-field description,[2,4] in which the effective magnetic field changes sign in the transition region, and using a bosonized Luttinger-liquid formalism, in which the interaction coefficient g is allowed to vary continuously within the transition region. We also show that the existence of an adiabatic junction between the two quantized Hall regions would allow construction of a dc step-up transformer, where the output voltage is larger than the input voltage supplied by the power source.

Consider the geometry illustrated in Fig. 1, where there is a narrow wire (or ''point contact'') connecting two macroscopic quantized Hall regions, with different electron densities corresponding to $\nu=1$ and $\frac{1}{3}$, respectively. We assume that each of the edges is sufficiently long that local thermal equilibrium is established on the edge at a voltage labeled V_j, where $j=1$ and 2 denote, respectively, the incoming and outgoing channels on the $\nu=1$ side of the junction, and $j=3$ and 4 denote the incoming and outgoing channels on the $\nu=\frac{1}{3}$ side. We also assume that the external current contacts are ''ideal,'' so V_1 and V_3 are equal to the voltages in the leads.[5]

If the voltages of the external leads are equal to each other, then the system will be in global thermal equilibrium, with all V_j being equal. More generally, if $e|V_1-V_3|$ is smaller than temperature T, the voltages V_2 and V_4 will be linear functions of V_1 and V_3, and we may write

$$V_2=\alpha V_1+(1-\alpha)V_3, \tag{1}$$

$$V_4=\beta V_1+(1-\beta)V_3, \tag{2}$$

where α and β depend on the characterisitics of the connecting junction, including T.

The current on edge j is given by $I_j=\nu_j V_j(e^2/h)$, and the energy flux along the edge is $I_j V_j/2$. Thus current conservation through the junction requires that

$$\beta=3(1-\alpha), \tag{3}$$

while the requirement that the outgoing power be equal to or less than the power incident on the junction implies

$$1/2\leqslant\alpha\leqslant 1. \tag{4}$$

The two limiting situations, where there is no energy loss in the junction region, are $\alpha=1$ and $\beta=0$, which corresponds to zero current transmission through the junction, and $\alpha=\frac{1}{2}$ and $\beta=\frac{3}{2}$, which is what we mean by an ''adiabatic junction.'' This regime was also found by Wen[6] who carried

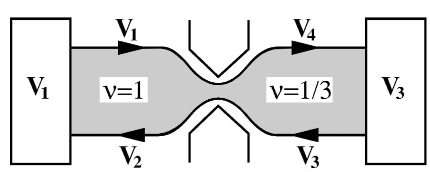

FIG. 1. Junction connecting quantum Hall states with different filling factors $\nu=1$ and $\frac{1}{3}$. The quantum point contact is produced by a narrow neck with the width of the order of the magnetic length. Arrows show the direction of the edge states.

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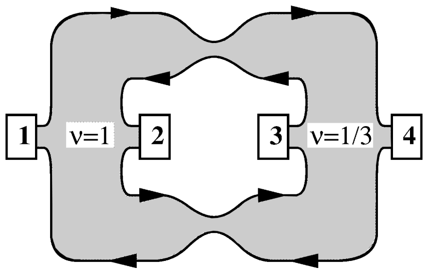

FIG. 2. Realization of the dc step-up transformer in the ring geometry. Two quantum point contacts separate regions with different filling factors. If a battery with voltage V is attached to contacts 1 and 2, then the voltage drop between 3 and 4 can be $3V$ in the limit of infinite load resistance.

out a similar analysis for a different set of Hall states. The more familiar case of a wide junction, where equilibration is established along a relatively long boundary separating bulk regions with $\nu=1$ and $\frac{1}{3}$, corresponds to parameters $\alpha=\frac{2}{3}$ and $\beta=1$, which is not dissipationless.

If we set $V_3=0$, and supply a small voltage V_1 to the other current lead, then a voltmeter connected between the opposite edges of the $\nu=\frac{1}{3}$ wire will measure the voltage $V_4=\beta V_1$. Moreover, the two-terminal conductance G, defined as the ratio between the current I in the leads and the input voltage V_1, is given by

$$G=\beta e^2/3h. \tag{5}$$

If we can construct a junction with $\beta>1$, then we can obtain a voltage V_4 which is larger than the input voltage, and we obtain $G>e^2/3h$. This last result violates the common belief that the two-terminal conductance of a quantum Hall system is always less than the bottleneck with lowest conductance, as the two-contact resistance of ideal leads connected to a single $\nu=\frac{1}{3}$ region would be $e^2/3h$. This also emphasizes an important point made by several authors that the question of conductance is subtle, and should be formulated with a definite experimental arrangement in mind.[7–9]

A more efficient voltage transformer may be realized with the ring geometry illustrated in Fig. 2. If a battery with voltage V is connected to ideal current contacts at points 1 and 2, and a load with resistance R is connected to points 3 and 4, then if the junctions between the regions of $\nu=1$ and $\frac{1}{3}$ are perfectly adiabatic ($\beta=\frac{3}{2}$), the voltage across the load resistor will be equal to $3V/(1+12h/e^2R)$. When $R=\infty$, this device draws no current from the battery, and the output voltage is $3V$. More generally, the output current is one-third of the input current. If $R\gg 12h/e^2$, the output voltage is close to $3V$, and the power lost in the transformer is small compared to the power delivered to the load.

To demonstrate the possibility of an adiabatic junction between $\nu=\frac{1}{3}$ and 1 states, we first use the fermion–Chern-Simons approach.[2,4] In the mean-field approximation the $\nu=\frac{1}{3}$ state is viewed as a completely filled Landau level for composite fermions. This also holds for the $\nu=1$ state except that the effective magnetic field is opposite to the direction of the external magnetic field. Therefore, a narrow wire at either filling factor with sufficiently sharp boundaries can be described in the Landau gauge by a single-energy band with two chiral edge channels. The two filling factors can be easily distinguished in a wire much wider than the magnetic length. In particular, the local electron density is three times greater in the $\nu=1$ state. However, when the width of the wire is of the order of the magnetic length the distinction between the two states disappears. Then the density is not a good way to differentiate between the states. In fact, on the mean-field level the two states look almost identical.

The transition between the two states can be carried out in the following way. On one side we have a wide $\nu=1$ state with a single energy band in the Landau gauge. The wire is then narrowed gradually on the scale of the magnetic length. When the width of the wire is of the order of the magnetic length, the energy spectrum is mainly determined by the confinement potential rather than the magnetic field. Therefore, reducing the effective magnetic field along the wire by reducing the density should not change radically the energy spectrum. Higher composite fermion energy bands corresponding to other fractions remain unfilled so that there is a single pair of edge channels. As the filling factor is reduced below $\frac{1}{2}$ the effective magnetic field changes sign and is slowly brought to its $\nu=\frac{1}{3}$ value. Then the wire is widened and represents a well-defined $\nu=\frac{1}{3}$ state.

Although the composite fermion analysis can be extended to find the chemical potentials of edge channels,[10] we take a different approach here. It has been argued by several authors[11,9] that a quantum wire with filling factor $\nu=1$ or $\frac{1}{3}$ can be modeled by a Luttinger Hamiltonian of the form

$$H=\frac{\hbar}{4\pi}\int_{-\infty}^{+\infty} v_F dx\left[\left(\frac{d\phi_L}{dx}\right)^2+\left(\frac{d\phi_R}{dx}\right)^2+\frac{g}{2}\left(\frac{d\phi_L}{dx}+\frac{d\phi_R}{dx}\right)^2\right]. \tag{6}$$

We define charge-density operators ρ_j by $d\phi_j/dx=2\pi\rho_j$, and we assume commutation relations

$$[\phi_j(x),\phi_{j'}(x')]=(-1)^j i\pi\,\mathrm{sgn}(x-x')\,\delta_{jj'}, \tag{7}$$

where $j=1$ and 2 refers to the indices R and L, respectively.

In the $\nu=1$ state the density operators ρ_j correspond to the actual electron density at a given edge, and $g=0$ for a sufficiently wide wire. In the $\nu=\frac{1}{3}$ state, however, $g=8$ and the Hamiltonian (6) can be diagonalized by making a Bogoliubov transformation to the fields $\tilde{\phi}_j$, which correspond to the actual electron density at a given edge,

$$\tilde{\phi}_L=\frac{1}{2}(1+1/\sqrt{1+g})\,\phi_L+\frac{1}{2}(1-1/\sqrt{1+g})\,\phi_R,$$

$$\tilde{\phi}_R=\frac{1}{2}(1-1/\sqrt{1+g})\,\phi_L+\frac{1}{2}(1+1/\sqrt{1+g})\,\phi_R, \tag{8}$$

and obey slightly different commutation relations:

$$[\tilde{\phi}_j(x),\tilde{\phi}_{j'}(x')]=(-1)^j i\pi\nu\,\mathrm{sgn}(x-x')\,\delta_{jj'}. \tag{9}$$

The general relation between g and the filling factor valid for the simplest fractions, with a single edge state, is

$$\nu=(1+g)^{-1/2}, \tag{10}$$

where ν^{-1} must be an odd integer.[3]

The validity of the Luttinger Hamiltonian (6) is based on the existence of the two chiral boson modes propagating in the opposite directions along the wire. Let us assume that conditions in the wire vary adiabatically (i.e., slowly on the scale of the magnetic length), in such a fashion that there are two running modes at any point in the transition region between the two quantum Hall states. Then we can describe the system by a Luttinger Hamiltonian (6) with $g=g_1$ for $x<-L/2$, $g=g_2$ for $x>L/2$, and g varying continuously from g_1 to g_2 for $-L/2<x<L/2$. (The effects of deviations from adiabaticity will be discussed below.)

To clarify the physics further, we note that according to the Luttinger-liquid theory, when the electron operators are expressed in terms of ϕ_L and ϕ_R, in the region where the wire is thin, one finds that the electron density in momentum space, $\langle n_k \rangle$ has singularities at all odd multiples of the Fermi momentum $k_F=\pi\rho$, where ρ is the density of electrons per unit length.[12] The amplitudes of the singularities all vanish rapidly when the strip becomes wide, however, except for the singularities at $k=\pm\nu^{-1}k_F$.[10]

By using commutation relations (7) with the Hamiltonian, we obtain the following equations of motion:

$$\frac{d\phi_L}{dt}=-v_F\left[\left(1+\frac{g}{2}\right)\frac{d\phi_L}{dx}+\frac{g}{2}\,\frac{d\phi_R}{dx}\right],$$

$$\frac{d\phi_R}{dt}=v_F\left[\left(1+\frac{g}{2}\right)\frac{d\phi_R}{dx}+\frac{g}{2}\,\frac{d\phi_L}{dx}\right], \tag{11}$$

where g and v_F are functions of x. The solution of these equations depends on the particular form of g. However, there are two limits when they can be solved exactly, independent of the way g varies in the transition region.[13] The first limit is when the wavelength λ of the incoming pulse is smaller that the length L of the transition region. In this case the solution can be found by making a Bogoliubov transformation to chiral modes (8), which correspond to density waves confined to a single edge. Thus in this limit there is no reflection from the junction.[13]

The other limit is when the wavelength λ of the incoming pulse is greater than the length L of the transition region. Then we can solve the problem separately in the two exterior regions, and apply the matching conditions that ϕ_j must be equal at the two sides of the junction. We formulate a scattering problem by forming an incoming wave with a current of unit amplitude from the filling factor ν_1 side. Then the current in the reflected wave is given by the reflection coefficent r and the transmitted wave by the transmission coefficient t. We find the current reflection and transmission coefficients[14,15]

$$t=2\nu_2/(\nu_1+\nu_2), \tag{12}$$

$$r=(\nu_1-\nu_2)/(\nu_1+\nu_2), \tag{13}$$

where ν_1 and ν_2 are related to g_1 and g_2 according to Eq. (10). It is easy to see that these coefficients satisfy the law of current conservation $r+t=1$, as well as the law of energy conservation. In fact the coefficients can be obtained from these two conditions. For the particular values $\nu_1=1$ and $\nu_2=\frac{1}{3}$, we find that the reflection coefficient is $\frac{1}{2}$. If the incoming wave originates from the filling factor $\frac{1}{3}$ side ($\nu_1=\frac{1}{3}$, $\nu_2=1$), the reflection coefficient is $-\frac{1}{2}$. Minus implies that the reflected pulse has the opposite sign of density. Our results are similar to a wave reflection in a classical string with an impedance discontinuity,[16] the impedance being the inverse of the filling factor.

Knowing the reflection coefficients for the currents also allows us to find edge-state chemical potentials on the two sides of the transition for dc transport. Let us send an infinite wavelength pulse from the $\nu=1$ side with a current such that the voltage on that edge is V_1 and a pulse from the $\nu=\frac{1}{3}$ with voltage V_3. Then the outgoing currents can be found from Eqs. (12) and (13). The voltages on the outgoing channels are seen to obey Eqs. (1) and (2), with $\alpha=\frac{1}{2}$ and $\beta=\frac{3}{2}$.

Next we consider the deviation from adiabaticity which may be present in a real system. An impurity, or an irregularity in the confining potential on the scale of magnetic length, at point x in the narrow-neck region can give rise to backscattering. This is reflected by adding to the Hamiltonian a term

$$H'=\gamma\exp[i\phi_L(x)+i\phi_R(x)]+\text{H.c.} \tag{14}$$

The phase of the coefficient γ will depend on the position x, and its magnitude will depend sensitively on the width of the strip at that point. The amplitude will be very small if x is in a wide region, as there will then be little overlap between the wave functions for states on opposite edges of the wire.

The resistance due to backscattering is proportional to $|\gamma|^2$, if $|\gamma|$ is small. According to the standard renormalization group analysis, however, for a wire of constant width, if $g>0$, the value of $|\gamma|$ will increase with decreasing energy scale. Specifically, for voltages sufficiently small so that one is in the linear regime, the backscattering resistance of a wire should vary as T^{-y}, with[3,17]

$$y=2-2/\sqrt{(1+g)}. \tag{15}$$

For the present situation, where g varies with x, if the temperature is sufficiently high that the thermal length scale $\hbar v_F/k_BT$ is small compared to the size L of the transition region, Eq. (15) still holds, with g evaluated at the position of the impurity. The value of y obtained in this way would be intermediate between the values $y=0$ and $\frac{4}{3}$, that correspond to uniform quantum Hall strips with $\nu=1$ and $\frac{1}{3}$, respectively. If the temperature is sufficiently low that the thermal length is large compared to L, however, then we find, from a normal-mode analysis,[10] that the exponent y becomes equal to 1, independent of the precise location x of the scatterer.[18]

In any case, we find that the adiabatic fixed point, where $\beta=\frac{3}{2}$ and there is no backscattering, is unstable, according to a Luttinger-liquid analysis, so that any nonzero value of $(\frac{3}{2}-\beta)$ will grow with decreasing temperature and voltage. Thus, to observe the effect of voltage amplification, one must fabricate a junction with a value of $(\frac{3}{2}-\beta)$ as close as possible to zero, and then make the measurement at a temperature which is not too low.

There are several difficulties standing in the way of the experimental implementation of the dc transformer. First, the quantum point contacts must be approximately a magnetic length wide yet adiabatic. Second, the edges of the $\nu=1$ and $\frac{1}{3}$ states must be sufficiently sharp to support only a single-edge channel.

In order to make the junction as close as possible to adiabatic, one would like to avoid any roughness in the confining potential, as well as impurities, which could lead to backscattering. One must also worry, however, about the possibility of an abrupt change in the electron density or its profile across the width of the wire that could occur due to a spontaneously formed domain wall, if the electron system goes through a first-order phase transition in the neck region.

Although we do not find any symmetry change between the $\nu=\frac{1}{3}$ and 1 states in a narrow wire, one cannot rule out the possibility of having several phases separated by first-order transitions. In fact, exact-diagonaliziation studies of systems with up to six electrons in a narrow wire suggest that there might be several distinct phases, separated by sharp transitions, between the densities which correspond to $\nu=1$ and $\frac{1}{3}$.[19,20] (The calculated states have different density profiles across the wire, corresponding roughly to phases with one, two, or three distinct rows of electrons.)

Even if there is a sharp transition in a long wire, however, it might be possible to obtain a smooth transition in a properly engineered point contact. Moreover, it is possible in principle to cancel the reflected amplitude from one density discontinuity with a wave reflected by a second discontinuity or by an impurity placed at an appropriate position, using destructive interference. Such a complicated structure may be difficult to achieve by design, but might occur naturally in some fraction of samples due to random flucuations during fabrication.

Shortly after this paper was originally submitted, a paper was posted[21] suggesting that conductance $G>e^2/3h$ could also occur in tunneling through a barrier between a $\nu=\frac{1}{3}$ edge and a three-dimensional electron gas. We believe that for a tunnel junction this would not occur at temperatures low enough for the model to be applicable, but that it might occur for a pinhole of proper size and shape.

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